Some Probability Problem Solving Techniques

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Table of Contents

Introduction

- We talk a lot about probability in the abstract in this club, but today we are going to look at some probability concepts that we can use to solve problems.
- There will be several problems throughout these slides, so feel free to come up if you want to work one out, or shout out your answers.
- These techniques may be useful for our Probability Game Night at the end of the semester.

Expected Value

- Discrete Case (Problems we'll do today):

$$\mathbb{E}[X] = \sum_{n=0}^{\infty} n \, p_X(n)$$

- Continuous/General Case:

$$\mathbb{E}[X] = \int_{\Omega} x \, f_X(x) d\Omega$$

Linearity of Expectation

- Linearity of Expectation essentially means that, if you have two random variables and you want to find the average of their sum, you can find their individual averages, and then add them:

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

- Ex: If the expected amount of rain on Saturday is 2 inches, and the expected amount of rain on Sunday is 1 inch, what is the expected amount of rain over the entire weekend?

Problems

- Suppose that N trick-or-treaters enter an elevator form the lobby of haunted hourse with M floors above them. Each one, uniformly at random, picks one of the M floors and presses the button for the floor. What is the expected number of floors the elevator stops at?

Problems

- Suppose that a child is going trick-or-treating, and wants to collect all n different types of candy. At each house they stop at, they are given one of the n candies, uniformly at random. What is the expected number of houses the child needs to stop at?

Random Walks

- Random walks are a pretty general and common type of problem in probability.
- Simple case: suppose that you are standing on a number line at 0 with a coin. If you flip heads, move forward, if you flip tails, move backwards.
- What is your expected position after a certain number of flips? What is the probability that you return to 0?

Problems

- Suppose a ghost starts at 0 on the number line and starts the random walk described on the last slide. What is the probability of it reaching 10 before -1?

Problems

- Suppose N people are sitting around a circular table. Person 1 starts with a giant bowl of candy. At each point in time, the person with the bowl eats a piece of candy, and then flips a coin. If it is heads, they pass it left. If it is tails, they pass it right. Who is the most likely to be the last to eat candy?

Problem

- Suppose I start at 0 with a coin. At each point in time, if I flip heads, I move forwards 1 space. If I flip tails, I move forwards 2 spaces. What is the expected value of the first position I am at past 100?
- What about if I do this with a 4 sided die? A 6 sided die?

Bayes Theorem

- The Theorem:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- In statistics, this theorem is used to do "Bayesian Inference", where we start with a 'prior' that encodes known information about a random variable X, and then when we observe a value from x, we update our prior to get a 'posterior'.
- Ex: Flipping a coin and trying to determine the probability of heads.

Monty Hall Problem

- You are on a game show. There are 3 doors in front of you, one with a car behind it, and two with goats behind it.
- You pick a door. Then, the host, knowing which door contains a car, opens a door with a goat behind it that you did not pick.
- You can either keep your choice or switch to the other closed door. What do you do?

Monty Hall Problem Variant

- Suppose that the setup is the same, however the host does not know what is behind each door. However, by coincidence, he opens a door at random, and there is a goat behind it.
- Should you switch?