PARADOXES

AIM OF WORKSHOP

WHY DO WE NEED PROOFS?

Does every statement need a proof?

Some statements seem pretty intuitive. Why not leave it at that?

How do I convince someone else my statement is true?

FORMAT OF WORKSHOP

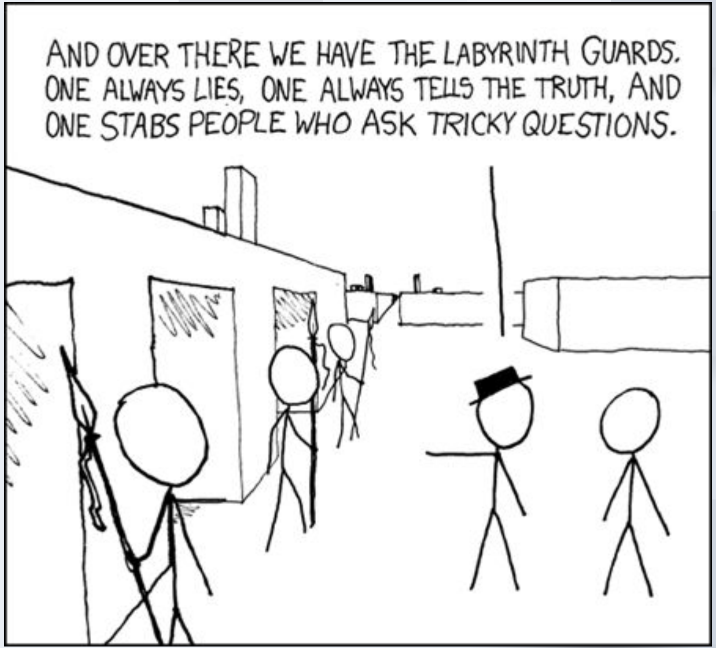
We'll walk through a few examples here.

Rest of the time will focus doing the workshop handout.

Go home and do the problems you can't solve here! Or read about the cool topics we name drop here!

Use the Slack to share interesting proofs / facts!

SELF REFERENCE



INTERESTING NUMBERS

CLASSIFYING NATURAL NUMBERS

- Consider the set of natural numbers = {1, 2, 3, … }
- Do you all think every natural number is interesting?
  - 1 is pretty interesting. It's the first one.
  - 1729 is pretty interesting. It's the smallest sum of 2 cubes in two different ways.
- With this we define a partition of the natural numbers into interesting and not interesting.

CLASSIFYING NATURAL NUMBERS

- Consider the set of not interesting numbers.
- There’s going to be a smallest not-interesting number.
- That’s kind of interesting…
- Contradiction! (?)

WHAT WENT WRONG?

- Formalize what “interesting” means.
- Maybe: “Appears in OEIS”? Then this is a ***well defined question***.

SET THEORY

RUSSELL'S PARADOX

DEFINE A SET

- In math, we'll talk about things. Things like
  - The integers.
  - Various shapes.
  - The even integers.
- Basically, we often talk about a group, a collection, a ***set*** of things.

“A set is a gathering together into a whole of definite, distinct objects of our perception [Anschauung] or of our thought—which are called elements of the set.” -- Georg Cantor

DEFINE A SET

- Definition 1: A set is a collection, a gathering of distinct things.
- You can describe sets with ***<u>set builder notation</u>***:
- A = { x : x is an even integer }
- B = { x : x is a person in this room }
- C = {x : x is a subset of A }

CAN WE PUT JUST ANYTHING AFTER THE ":"?

- You can describe sets with set builder notation:
- We call a set, D, ***<u>normal</u>*** if the set D is not an element of itself
  - The set of people in this room is a normal set
  - The set {A, B, C, {A, B, C}} is (normal/not normal)?
  - Can someone try describing a **not** normal set?

CAN WE PUT JUST ANYTHING AFTER THE ":"?

- We call a set, D, ***normal*** if the set D is not an element of itself.
- Consider the set of sets
  - N = { x : x is normal }
- Is N normal?

CONTRADICTION

WHAT?? WHAT WENT WRONG? :((((((

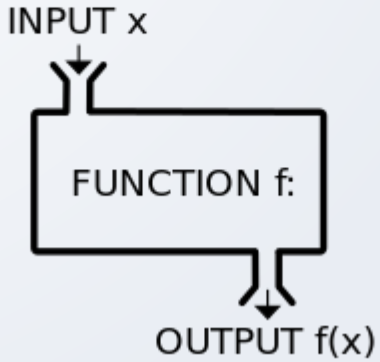
CAN WE PUT JUST ANYTHING AFTER THE ":"?

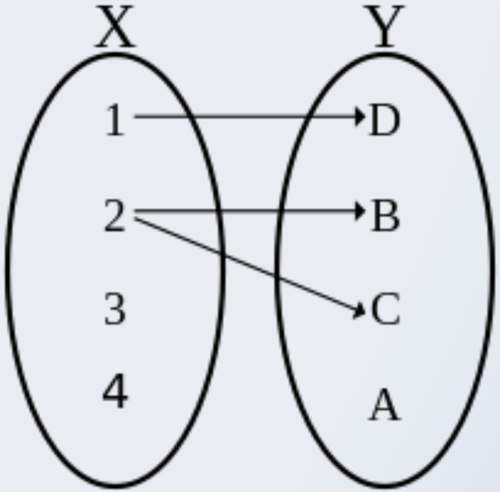
- Consider the set of sets
  - N = { x : x is normal }
- We cannot put anything we want after the :
- Can we make set theory axioms which don't run into paradoxes?
- https://www.scientificamerican.com/article/what-is-russells-paradox/

COUNTING

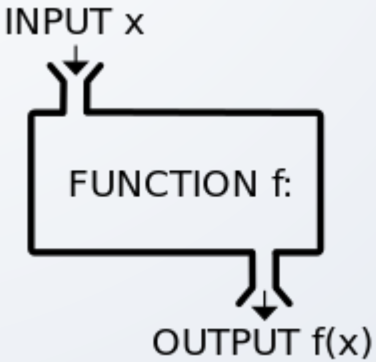
THE REAL NUMBERS

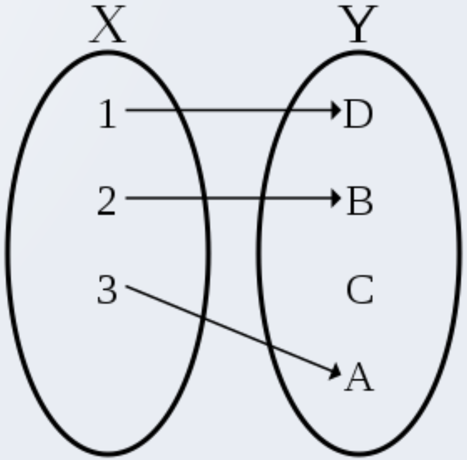
FUNCTIONS: CRASH COURSE

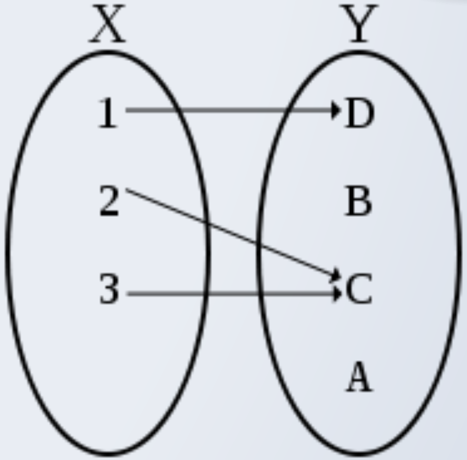




FUNCTIONS: INJECTION/1 to 1

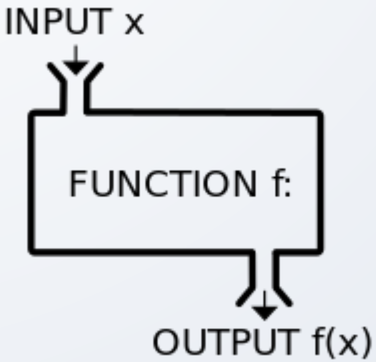


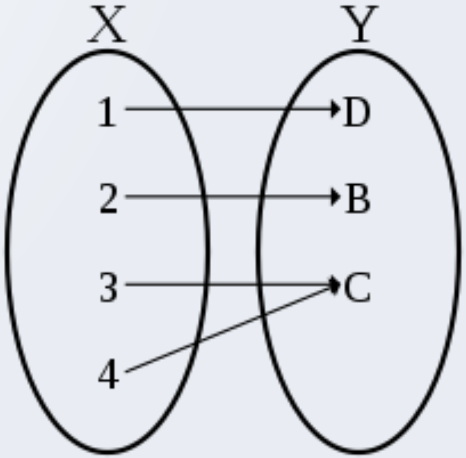


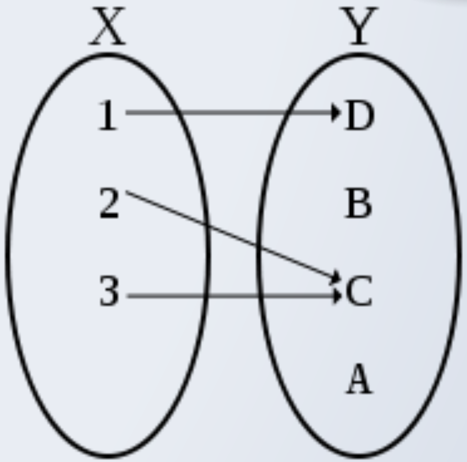


DEF: A function f is ***<u>injective</u>*** if for every element in Y there is *<u>at most</u> 1 element in X* that maps to it

FUNCTIONS: SURJECTIVE/ONTO

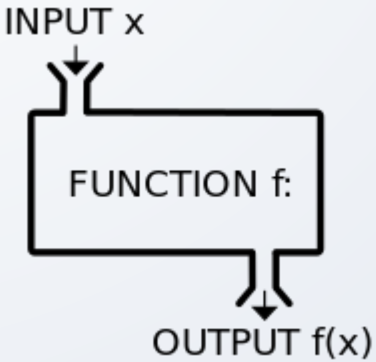


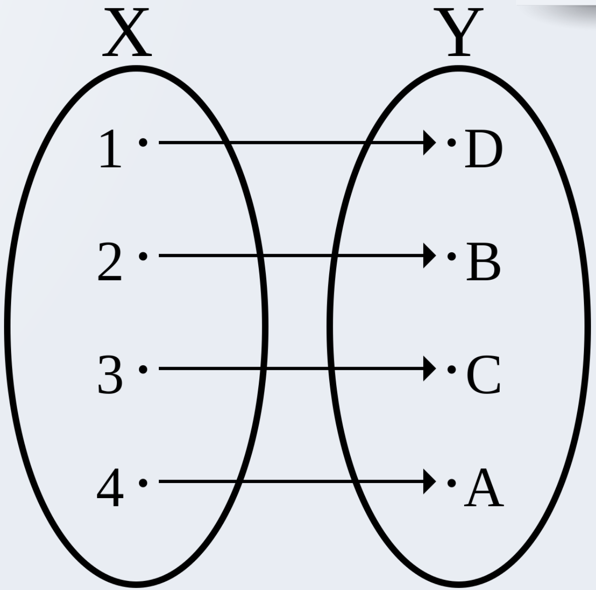


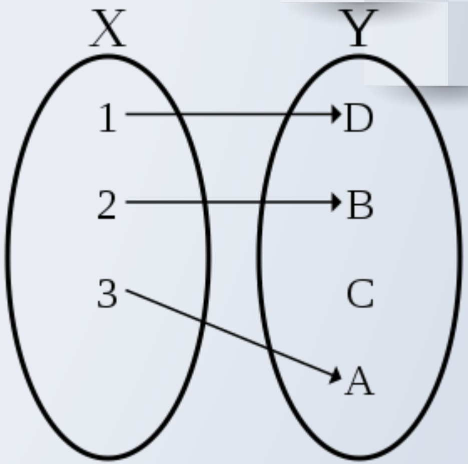


DEF: A function f is <u>***surjective***</u> if for every element in Y there is *<u>at least 1</u> element in X* that maps to it

FUNCTIONS: BIJECTION







DEF: A function f is ***<u>bijective</u>*** if for every element in Y there is <u>*exactly*</u> *1 element in X* that maps to it

DEFINE THE SIZE OF A SET

Any guesses?

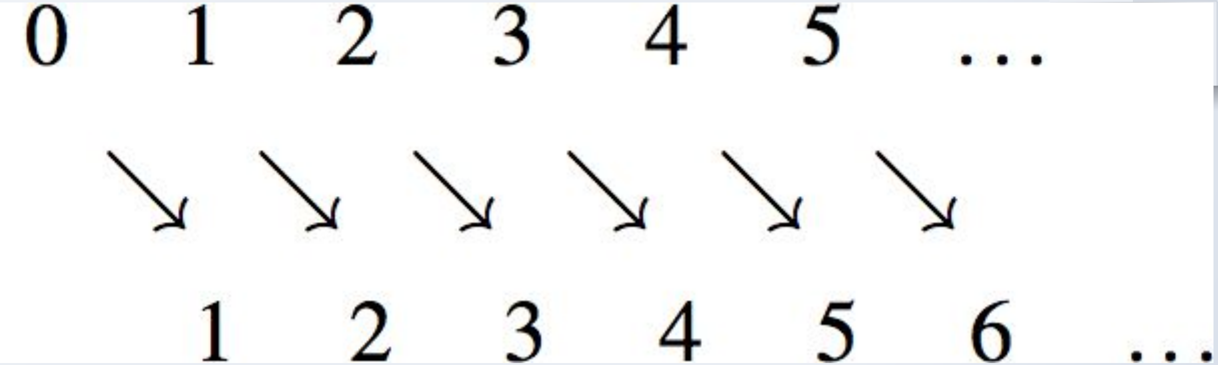
A set A has ***size*** n if there exists a bijection from A to the numbers {1, 2, 3, ..., n}

THINGS ARE INTERESTING WITH INFINITE SETS

A = {0, 1, 2, 3, … }

B = {1, 2, 3, … }

Are A and B the same size?



A = {x : x is an integer}

B = {x: x is an even integer}

LET’S COUNT THE REAL NUMBERS

What’s the cardinality of the real numbers?

Let’s make a bijection to the natural numbers.

Say there exists a bijection to the natural numbers.

CANTOR'S DIAGONALIZATION ARGUMENT

Say we have a way to enumerate the real numbers

$$
\begin{array}{|llllllllllllll|}
\hline
s_1 & = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
s_2 & = & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
s_3 & = & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\
s_4 & = & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots \\
s_5 & = & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \dots \\
s_6 & = & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & \dots \\
s_7 & = & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\
s_8 & = & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & \dots \\
s_9 & = & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & \dots \\
s_{10} & = & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & \dots \\
s_{11} & = & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\hline
\end{array}
$$

$$
\begin{array}{|llllllllllllll|}
\hline
s & = & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & \dots \\
\hline
\end{array}
$$

s differs from s_n in the nth spot. So where does s go in this list?

CONTRADICTION!!!!

The real numbers does not have cardinality equal to the natural numbers.

There are **multiple types of infinity!**

Look up “Continuum Hypothesis”

Thanks to Wikipedia for all the pictures!

TOMORROW'S WORKSHOP:

FORMAL INTRODUCTION TO PROOFS

KLAUS 2447 @ 6:30 P.M

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