INTRODUCTION TO THEORY CS: WORKSHOP 1

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1. Workshop Problems

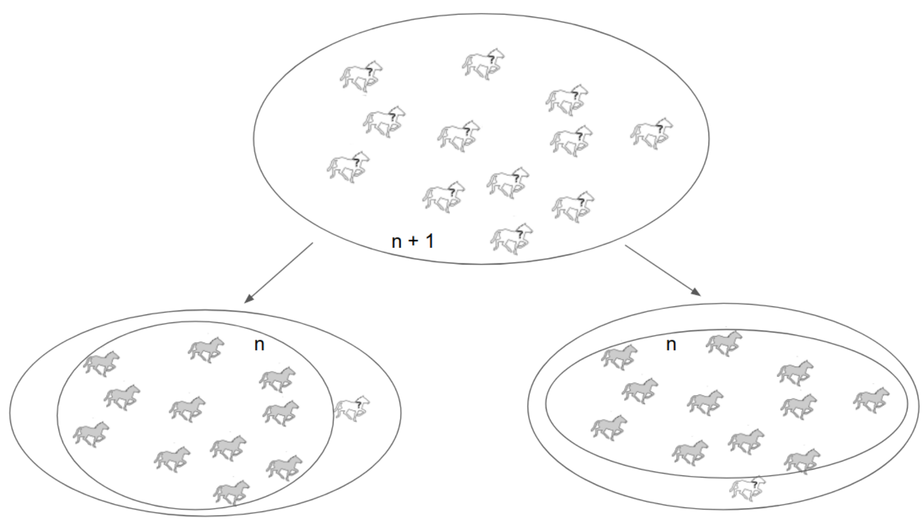
(1) **Induction** I have a set S with the following two properties.
   - 1 is in my set S.
   - If n is in my set, $n+1$ is in my set.

   List some numbers that are in S. What is the set S?

(2) Prove that for $n \geq 2$, a $2^n \times 2^n$ board with a corner square removed can be covered by L tiles of size 3. What is my set S?

(3) Prove that
$$\sum_{i=1}^{n}(2n-1) = n^2.$$
   You can do it algebraically, or geometrically.

(4) **All horses are the same color!** I am going to show that in any set of n horses, all of them are the same color.
   - When we have 1 horse, it's pretty trivial
   - Next we prove: If any set of n horses, all are the same color, we would like to say this implies the same for any set of $n+1$.

     Say we have $n+1$ horses. Then, take some subset of n of them. These must all be the same color. But we left one horse out! Just consider another subset of n which includes the odd horse out. These n must also all be the same color. We have accounted for all horses, so all must be the same color.
   - So uh...are all horses the same color?

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(5) **Bertrand's Box Paradox** There are three boxes:
- a box containing two gold coins,
- a box containing two silver coins,
- a box containing one gold coin and one silver coin.

We choose a box at random, and one of its coins at random and it turns out to be a gold coin! What is the probability that the next coin you pick is gold?

(6) **Potatoes** Fred brings home 100 pounds of potatoes, which (being purely mathematical potatoes) consist of 99 percent water. He then leaves them outside overnight so that they consist of 98 percent water. What is their new weight? (From The *The Universal Book of Mathematics*)

(7) **Counting** Find the cardinality of $2^{\mathbb{N}}$.

(8) **The Birthday Problem** What is the probability that two people in a room of n people share the same birthday?
   (a) Intuitively, what do you guess the probability is?
   (b) How many people do you need to guarantee two people have the same birthday?
   (c) Come up with a formula dependent on n. What is the limiting behavior? What value of n do you need to hit a 50 percent probability?

2. Interesting Facts to Consider

2.1. **Banach-Tarski Paradox.** We have a solid unit ball in $\mathbb{R}^3$. We can decompose the ball into a finite number of disjoint subsets and then use these pieces to make two solid unit balls in $\mathbb{R}^3$. Somehow, we've taken an object of volume 1, broken it into pieces, and created two objects of volume 1! It turns out that this paradox exists because of the **axiom of choice** - some of the pieces we break our ball into are "non-measurable".

2.2. **Halting Problem.** Suppose we have a rigorous "model of computation" and some computational machine which allows us to write programs. These programs can take some input, and output yes/no answers ("decision problems"). E.g., you might have a program which takes in a list of integers, and outputs yes/no to tell whether the list is sorted.

Can we write a "correct" program which takes as input another program P as well as a sample input x, and outputs yes/no to tell whether P will halt when run on input x? "Correct" here means the program always outputs the correct answer, and always halts. It turns out the answer is no. To read more see the **halting problem**.