WORKSHOP 2 FALL 2019
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INTRODUCTION TO THEORY CS: PROOFS

1. Prove that $6|(7^n - 1)$ for all positive integers n. (From The Art and Craft of Problem Solving)
2. Prove that $1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2$ (from AoPS).
3. Every positive integer greater than 1 is a product of primes.
4. Prove that the nth fibonacci number F_n is at most 2^n
5. We say that $f = O(g)$ if there exists an integer N and a constant c such that
$$f(n) \leq c \cdot g(n)$$
for all $n \geq N$. Prove
   - $n^3 + n^2 = O(n^3)$
   - $\sum_{k=1}^{n} k^c = O(n^{c+1})$ for any constant $c > 0$.
   - $n^{c+1} = O(\sum_{k=1}^{n} k^c)$ for any constant $c > 0$.
6. Suppose that B is such that $AB = BA = A$ for all matrices $A \in \mathbb{R}^{n \times n}$ then $B = I$.
7. (a) Is the product of two irrational numbers irrational?
(b) Is the sum of two irrational numbers irrational?
(c) Is an irrational number raised to an irrational number also an irrational number?
8. Prove the Pigeonhole Principle: "if n items are put into m containers, with $n > m$, then at least one container must contain more than one item." (Stolen from Wikipedia)
9. McDonalds only sells chicken nuggets in packs of 6 and 7. What is the biggest number of nugs that cannot be ordered. Prove that no number bigger than this can be done.
10. Let a_0 and b_0 be integers, and consider the following algorithm (Euclid's algorithm). Start with $i = 0$ and repeat:
    1. Find some integers q and r with $r < b$ such that $a_i = qb_i + r$
    2. Set $a_{i+1} = b_i$ and $b_{i+1} = r$.
    3. Repeat until $b_i = 0$. Output a_i.

    Answer the following two questions:
    1. Prove that the output of this algorithm is the greatest common divisor of a_0 and b_0.
    2. Prove the algorithm runs in $O(\log b_0)$ time (assuming basic operations take constant time).
11. Prove that the set of all finite subsets of the natural numbers is countable.
12. If five points are on a sphere, then some four are on a hemisphere.
13. Prove that for any $a, b \neq 0 \mod p$ and any c we have that there exist x, y such that $ax^2 + by^2 = c \mod p$.
14. Prove that there are an infinite amount of primes. **Hint:** Try proof by contradiction.
15. Prove that there are an infinite amount of primes of the form $4k + 3$.
16. (Cauchy Induction) Prove the AM-GM inequality if $a_1, ..., a_n$ are positive reals then
$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$$