Big O and Memesorts:

How slow can we go?

Neil Thistlethwaite

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However, it's often hard (and unnecessary) to characterize runtime and memory usage explicitly in terms of n – is there something better we can use?

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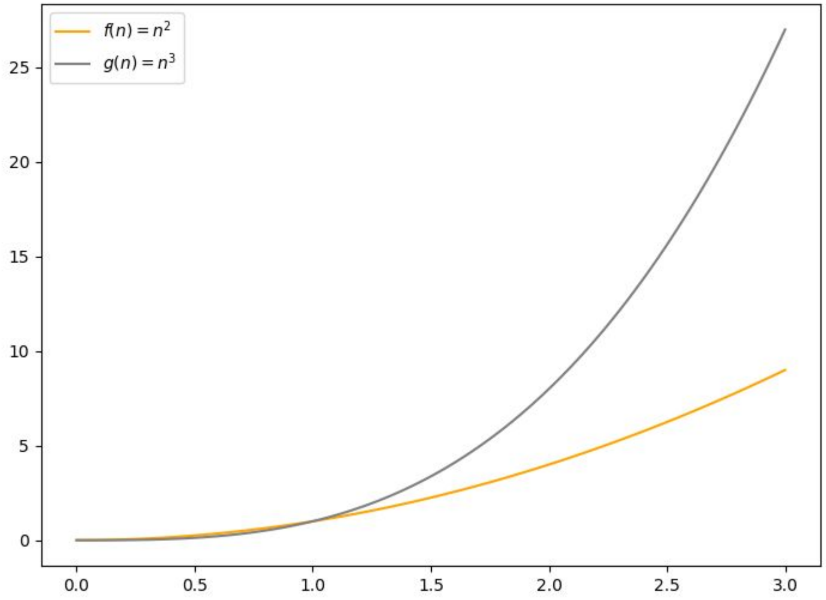
A pair of values C, n_0 that show this for some f and g are said to be **witnesses**.

Big O - Examples

$n^2 = O(n^3)$



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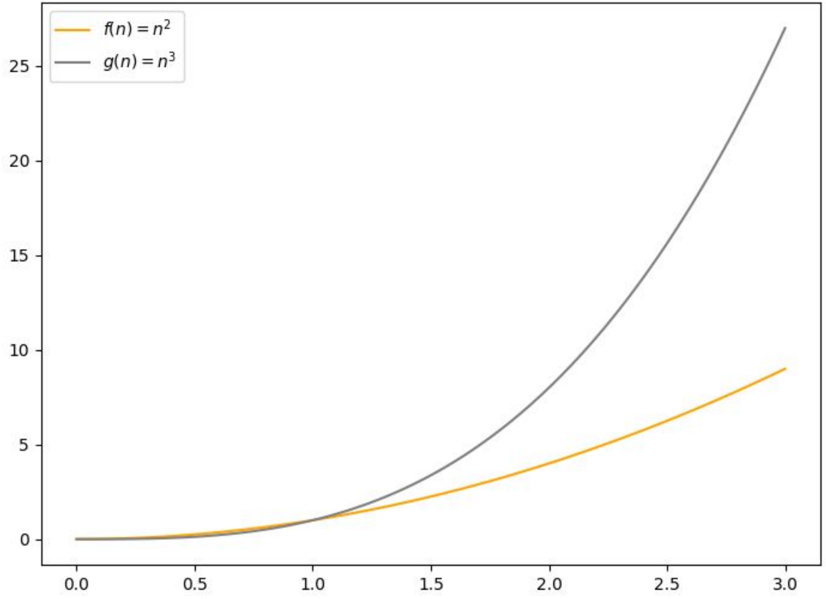

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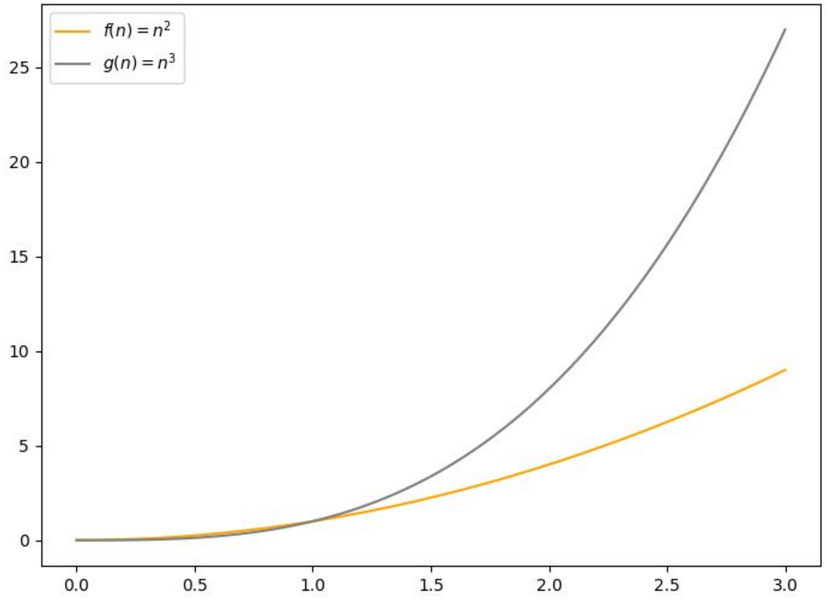

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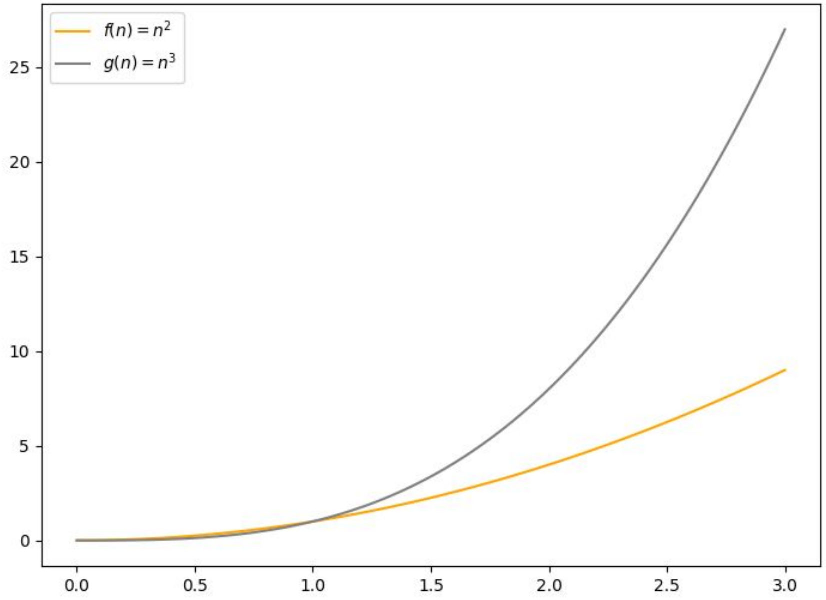

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e.g. let $f(n)$ represent the time it takes me to make n slides for this presentation. We can think of Sherry saying "Neil make slides" as the algorithm, and n as the input (the number of slides needed for Sherry to be happy with the presentation).

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Since each slide will take Neil roughly the same amount of time to make (constant time is $O(1)$), it’s going to take him $n * O(1) = O(n)$ time to make n slides.

Big O - Sorting

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NO!

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Suppose we're sorting a list $[a_1, a_2, \ldots , a_n]$. A single comparison between elements i and j returns the answer to whether $a_i > a_j$. Observe the following: without knowing anything about the elements, there are n! possible permutations, any of which could be the possible correct sorted list (why?).

Sorting Lower Bound Proof

Whenever we observe a comparison, we can at most rule out half of the remaining possible permutations. This is because if any given comparison were to rule out significantly more than half of the permutations, we could “adversarially” choose to give the opposite result for that comparison.

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Therefore, our “worst case” is actually when a comparison rules out exactly half of the remaining permutations. From here, simple math will give us the number of comparisons we need.

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By definition of big O,

$$t = O(n \log_2 n) - O(n) + O(\log_2 n) = O(n \log n)$$

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> This is left as an exercise to the reader.

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Sorting as a Service (SaaS): It's 2019! Why sort yourself, when you can just send all your data to a sketchy API and have them sort it for you?

Problems! (Courtesy of Sherry)

1. Find the big O of $\sum_{k=1}^{n} k^c$ for some $c > 1$
2. Find the big O of $\sum_{k=1}^{n} \log k$
3. Prove that $\log^k n = O(n^m)$ for any $k, m \in \mathbb{R}^{++}$
4. An algorithm flips a fair coin until it gets heads. What is the expected run time of this algorithm?
5. Answer the following two questions:
   (a) Prove that the output of this algorithm is the greatest common divisor of a_0 and b_0.
   (b) Prove the algorithm runs in $O(\log b_0)$ time (assuming basic operations take constant time).