Randomized Algorithms

Album 1 - Basic Probability

A **discrete random variable** is a function X from an outcome space Ω to $\mathbb{R}$. The **probability mass function** of a discrete random variable is a function which indicates the probability a discrete random variable takes on a certain value. For example, a dice throw is a discrete random variable (each face of the dice is mapped to a number from 1 through 6). The associated probability mass function, f, is $f(1) = f(2) = \ldots = f(6) = \frac{1}{6}$. A coin flip (also called a **Bernoulli** random variable, is a discrete random variable which maps Heads to 1 and Tails to 0. The associated PMF is $f(0) = f(1) = \frac{1}{2}$.

The **expectation** of a random variable X is

$$\mathbb{E}[X] = \sum_x x \cdot \Pr X = x$$

where the sum is over the set of values the random variable can take on. For example, the expected value of a fair die roll is

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2} = 3.5$$

Track 1. Prove that expectation is **linear**. That is, for two random variables X and Y (on the same outcome space Ω):

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Track 2. Say we can draw from a probability distribution of a random variable X. Let us call these draws $X_1, X_2, \ldots, X_n$. Let $Y = \frac{\sum_{i=1}^n X_i}{n}$. What is the expected value of Y? What does this tell us about how to estimate the mean of a random variable?

Album 2 - Randomized Algorithms Questions

Track 1. Given a fair six sided die create an random number generator for integers from 1 through 10.

Track 2. Given a biased coin that is heads with probability p where $p \in (0, 1)$ is an unknown parameter, design a protocol to get a fair coin flip. Analyze the expected number of coin flips that your protocol uses.

Track 3. There are n types of Kidz Bop CDs at your local target and for nostalgia you want to collect them all. Your local Target dispensary gives them out uniformly at random one at a time for a dollar. How much do you expect to spend before you have one of each type?

Track 4. Given two numbers a and b that are n-bits numbers and a random prime from the set of the first $3n$ primes. Bound from above the chance that $A \equiv B \mod p$ given that $a \neq b$.

Track 5. A series of numbers are coming in a stream, one at a time. You want to sample a random element from the stream. You don't know how long the stream is or anything about the numbers coming in. Design a protocol for doing this.

Track 6. You are given algorithm that succeeds with probability $p \in (0, 1)$ and distinct runs of the algorithm are independent. Fortunately, you can check if the algorithm succeeded. You keep running the program until you succeed. Bound from above the probability that you have to run the algorithm n times?