Counting Locally Flat-foldable Origami Configurations via 3-Coloring Planar Graphs

Alvin Chiu, William Hoganson, Sylvia Wu

An Introduction to Origami

What is Origami?

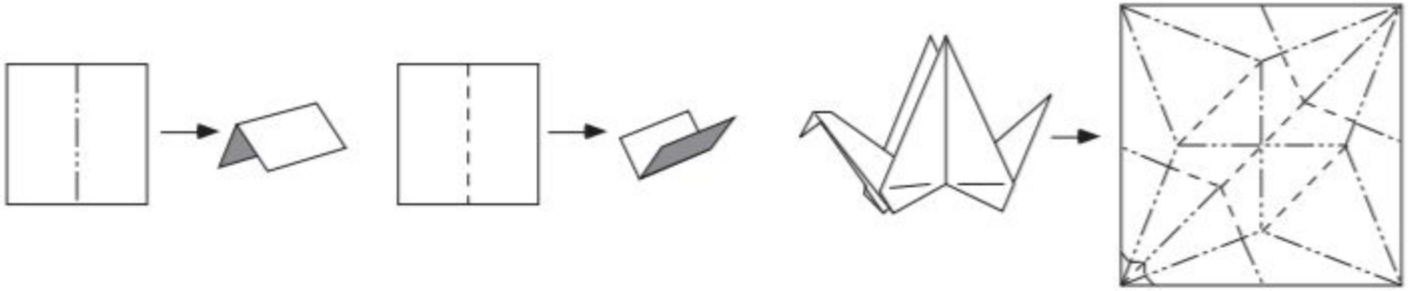
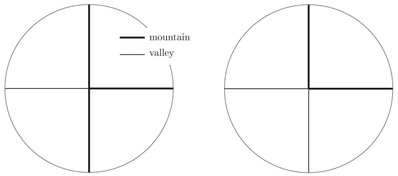


Figure 1: Mountain creases, valley creases, and the crease pattern for the flapping bird with MV assignment shown.

Mountain-Valley (MV) Assignments

- Assign values to creases to tell us how to fold our crease pattern
  - Mountain → 1
  - Valley → -1
- (Globally) valid: we can physically fold flat this MV assignment
- Locally valid: each vertex can be folded flat in an MV assignment

Maekawa's Theorem states that if a particular MV-assignment of the creases about a vertex v in G is valid (folds flat), the difference between the number of mountain creases and valley creases must be ± 2.

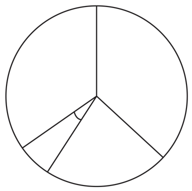


Flat Foldability

Let C be a single-vertex crease pattern with vertex v, and angles a_1, a_2, ..., a_{2n} be the angles surround it in clockwise order. ***Kawasaki's Theorem*** states that if C is flat foldable iff $a_1 - a_2 + a_3 - \ldots - a_{2n} = 0$.

Generalized Big-Little-Big

- If there are an odd number of consecutive smallest angles, $M - V = 0$ for the creases surrounding them.
- If there are an even number of consecutive smallest angles, $M - V = \pm 1$ for the creases surrounding them.
  - The two creases surrounding an angle between two bigger angles has to sum to 0, thus it is either ***M, V*** or ***V, M***.



Current Results: Counting Valid MV assignments

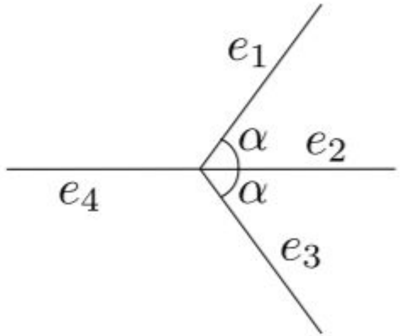
- Counting **globally valid** MV-assignments is an open problem.
  - Given a crease pattern, deciding whether it is flat foldable is strongly NP-hard.
  - Given an MV assignment to a crease pattern, deciding whether that is flat foldable is also NP-hard!

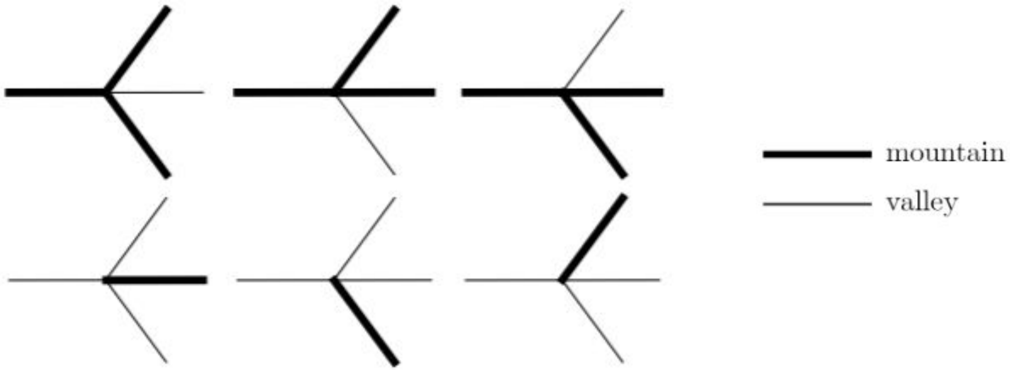
Our Goal: Counting Locally Valid MV assignments

- (Hull 2002) For a single vertex crease pattern with degree $2n$, there is an linear-time algorithm to count the number of locally valid MV assignments, using a recursive formula.
- (Bern and Hayes 1996) For a general crease pattern, there is a linear-time algorithm to find a locally valid MV-assignment (if possible).
- For larger crease patterns (with multiple vertices), counting the number of locally valid MV assignments is open.

How do we count all locally-valid MV assignments?

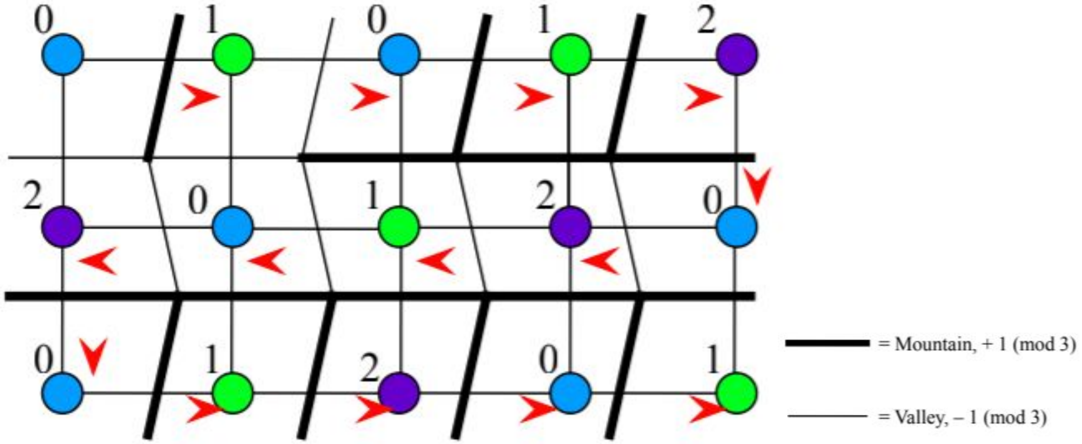
- Bird's Foot: 6 valid MV assignments
  - Maekawa's Theorem
    - 3 mountains and 1 valley, or 3 valleys and 1 mountain
  - Big-Little-Big
    - $e_1 + e_2 + e_3$ cannot all be mountain or valley.





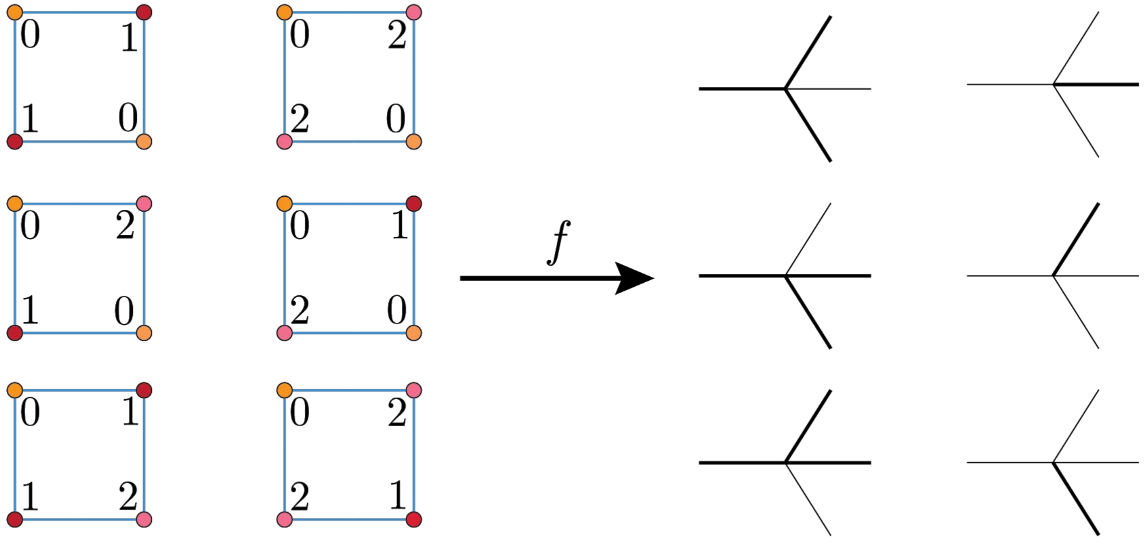
The Miura-Ori Bijection

- Locally Valid MV assignments of the Miura-Ori Crease Pattern ⇔ proper 3-colorings of a square grid with one vertex pre-colored
    - Shown in a paper by Hull and Ginepro (2014), which served as the starting point for our work.



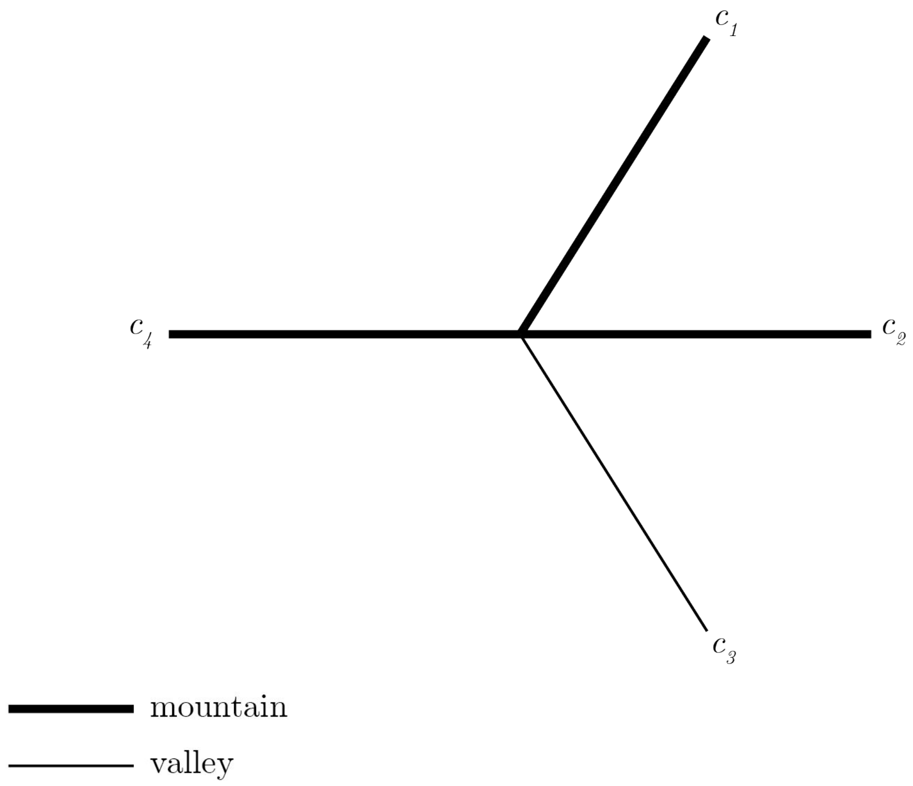
SAW Graphs: An Introduction

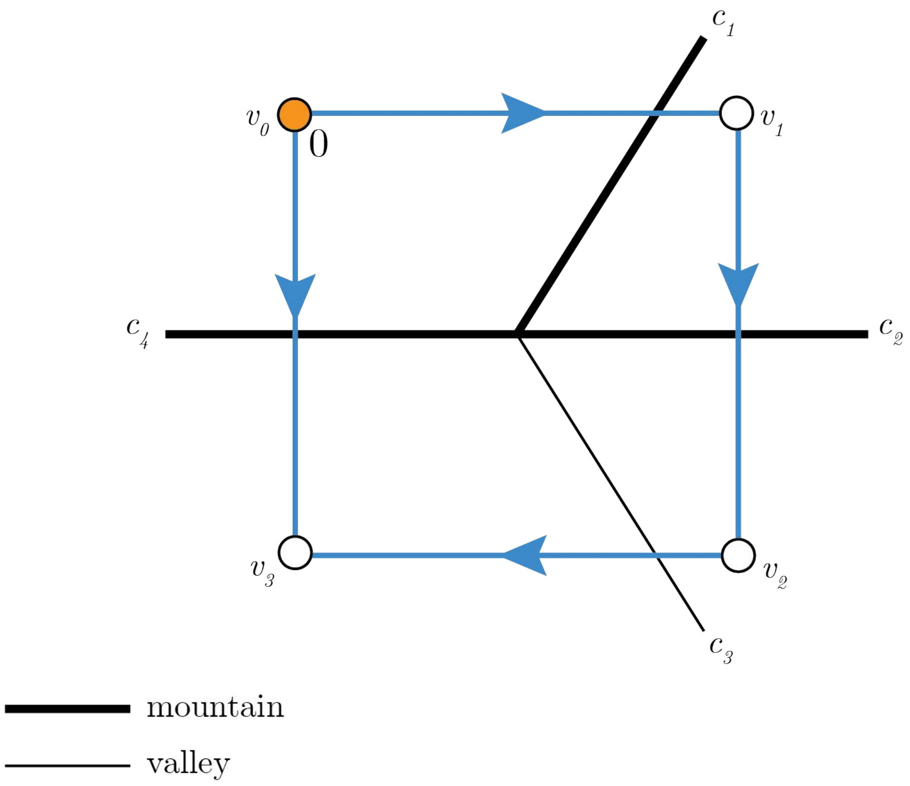
For our single-vertex crease pattern G, we create graph SAW(G) (blue below). We define a function $f : S \to M$ where S is the set of all proper 3-colorings of SAW(G) and M is the set of all valid MV-assignments of G.



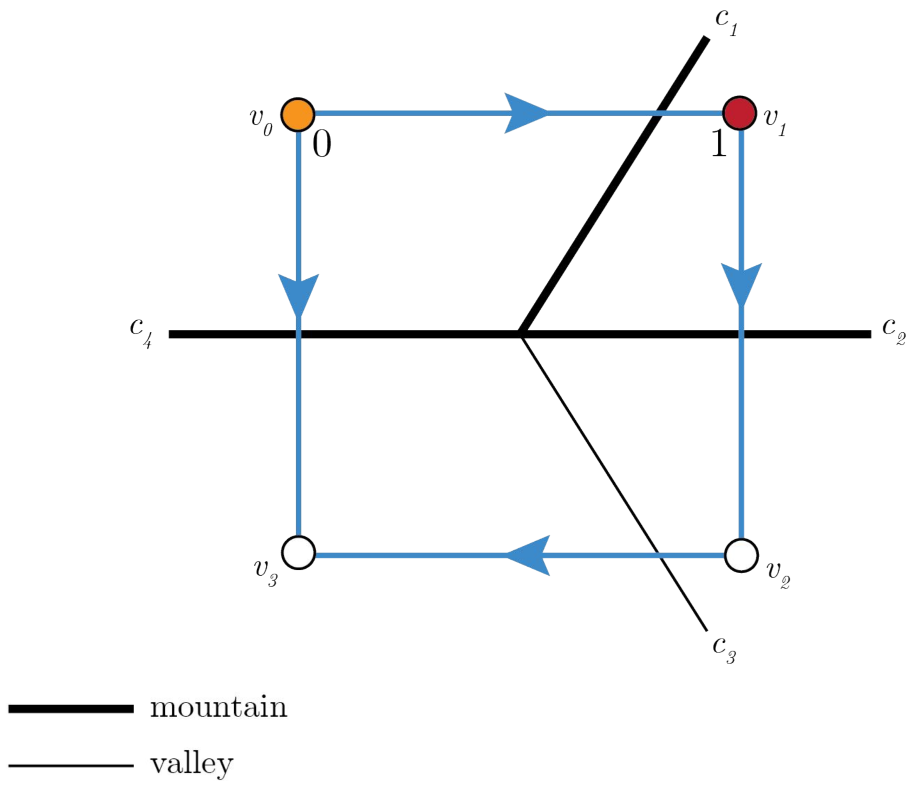
Proper 3-colorings of SAW(G)

Valid MV-assignments of G

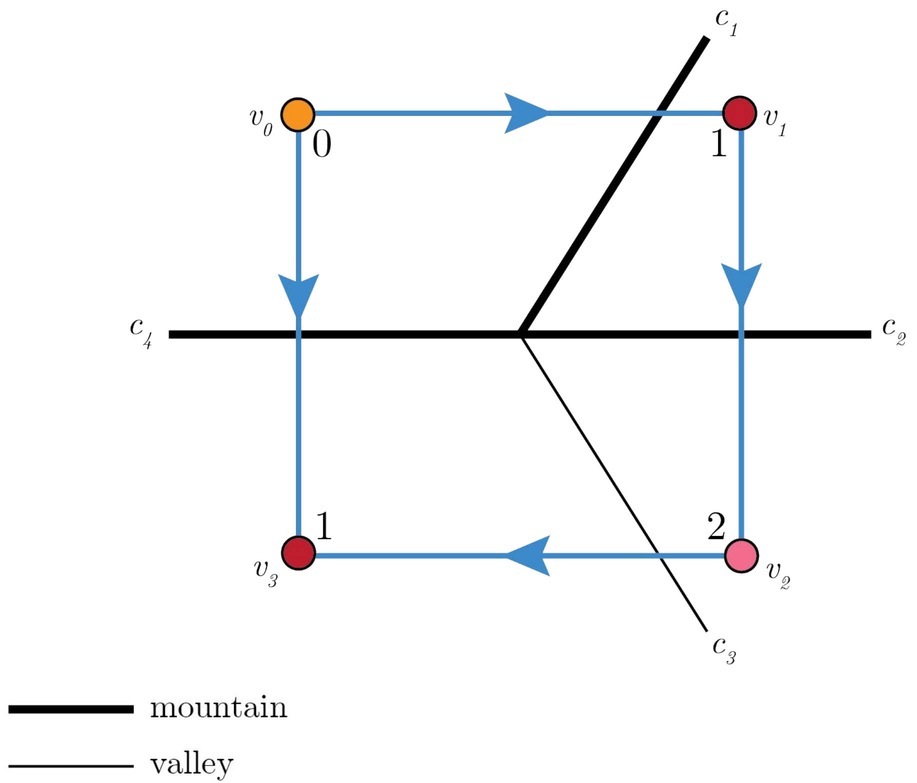

c_1
c_4
c_2
c_3
mountain
valley



c_1
v_0
0
v_1
c_4
c_2
v_3
v_2
c_3
mountain
valley



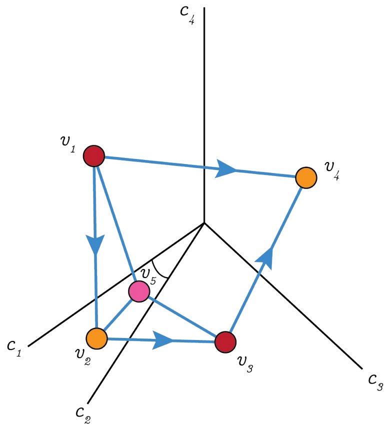
c_1
v_0
v_1
0
1
c_4
c_2
v_3
v_2
c_3
mountain
valley

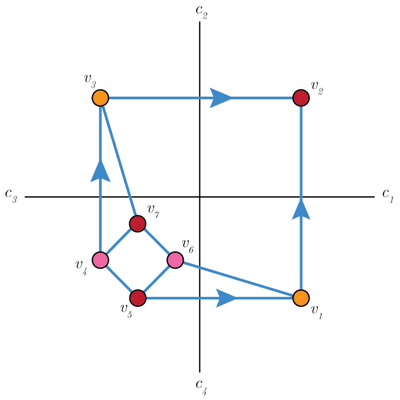


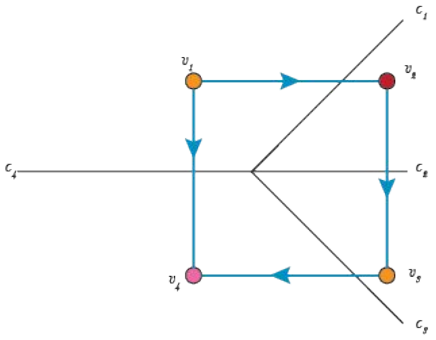
c_1
v_0
0
1
v_1
c_4
c_2
1
2
v_3
v_2
c_3
mountain
valley

SAW Graphs: All Degree 4 Vertices

- Graphs depend on crease pattern

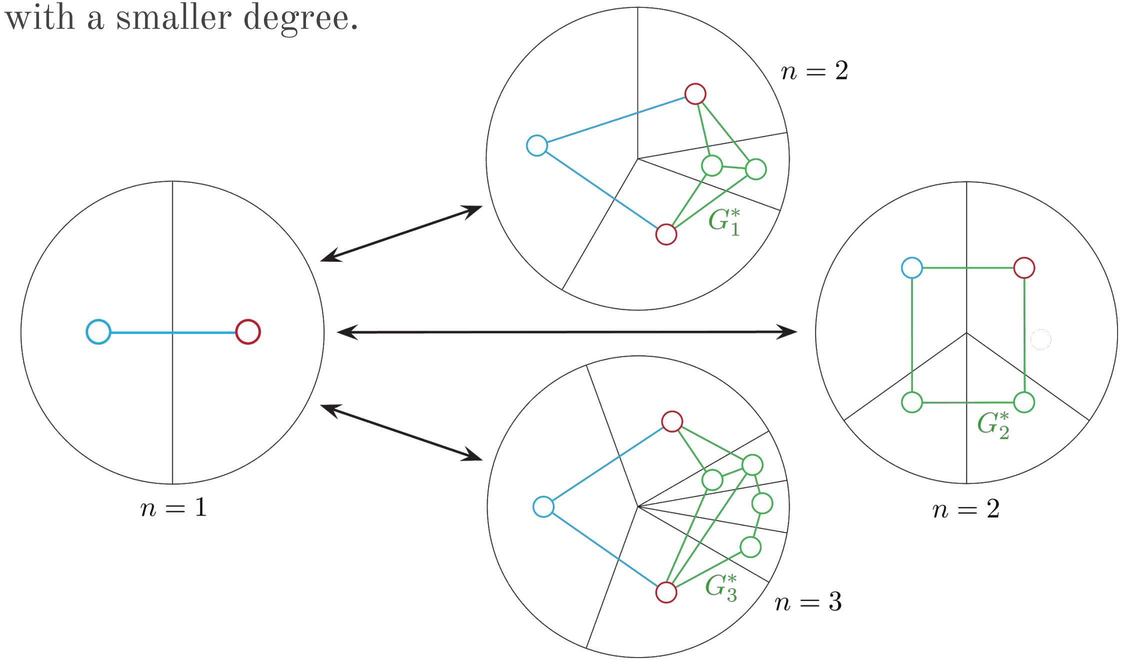






Recursion for Degree *2n* Vertices

- Can we find SAW graphs for any degree *2n* single-vertex crease pattern?
- Process: fold in the creases surrounding the smallest angles.
    - Results in a vertex with a smaller degree.



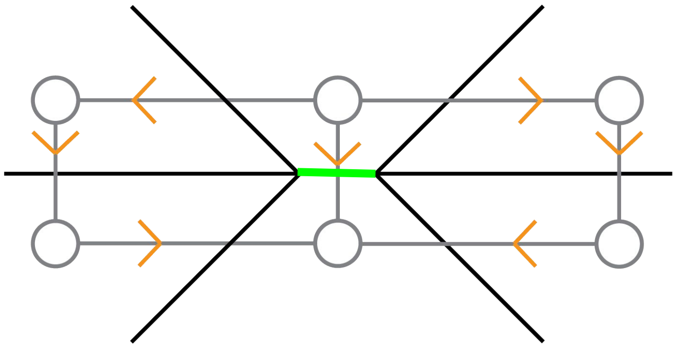
Nice Vertices with SAW Graphs

Definition
We define a single-vertex, flat-foldable crease pattern C to be ***k*-nice** if there are at most k consecutive smallest angles over all single-vertex crease patterns in the recursive folding of C.

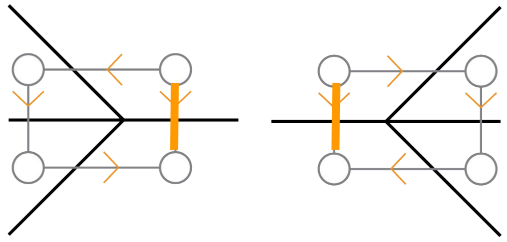
Theorem
For any ***3*-nice**, single-vertex, flat-foldable crease pattern C, there exists a SAW graph C^*.

Linking SAW Graphs

- If two vertices share a crease, then the directed edges in the two respective SAW graphs that crosses that common crease are merged if they have the same orientation.
  - The resulting graph will be the SAW graph for the larger crease pattern.

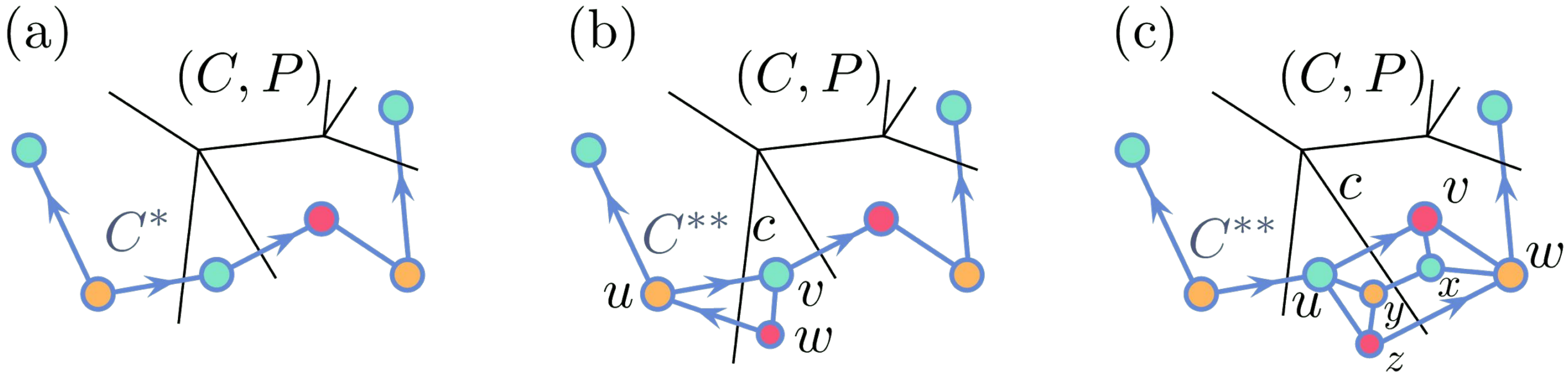


Gadgets to piece together



Modifying SAW Graphs

- SAW Graphs are **not unique**.
- There are two “gadgets”, the triangle and the triangular prism, which change the SAW graph but preserve the number of proper 3-colorings that it has.



Crease Patterns with SAW Graphs

Theorem

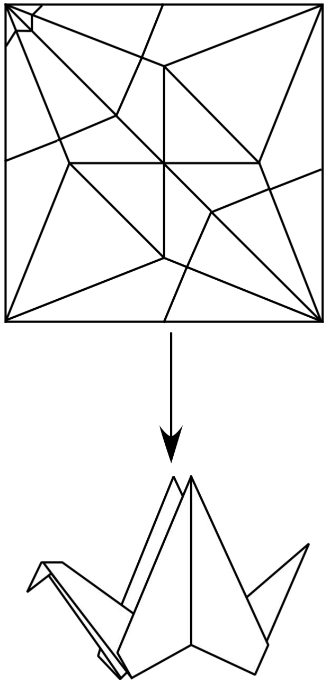
Let C be a crease pattern such that for all vertices $v \in V(C)$, there exists a SAW graph $V*$ where V is the single-vertex crease pattern corresponding to v. Then, there exists a SAW graph for the whole crease pattern C.

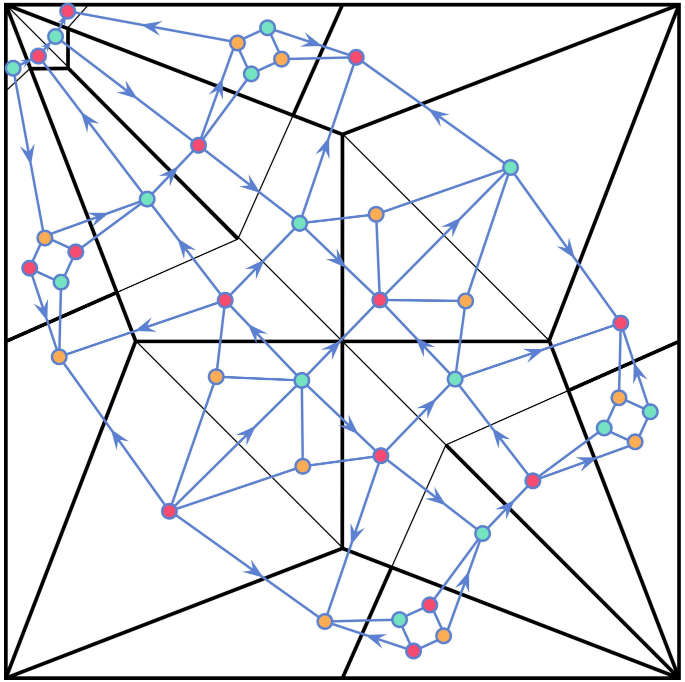
Proof

By induction!

The Origami Crane!

There are 93,312 ways that the crane locally folds flat!





Future Work

- Can we find SAW graphs for all single-vertex crease patterns?
  - Currently only know this for 3-nice vertices.

Questions?

- This material is based upon work supported by the National Science Foundation under Grant Number DMS-1851842. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Linear time algorithm for finding Locally valid MV assignment [Bern and Hayes 1996]

http://courses.csail.mit.edu/6.849/fall12/lectures/L03.html?notes=6&images=1